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One particle quantum equation in a de Sitter spacetime

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Abstract We consider a free particle in a de Sitter spacetime. We use a picture in which the analogs of the Schrödinger operators of the particle are independent of both the time and the space coordinates. These operators induce operators which are related to Killing vectors of the de Sitter spacetime.

1 Introduction

In this paper we use a generalized Schrödinger (GS) picture to describe a particle in a de Sitter spacetime. The GS picture is based on the principal series of the infinite-dimensional unitary irreducible representations (UIR) of he Lorentz group [1] and a spacetime transformation. The principal series correspond to the eigenvalues $1+\alpha^2-\lambda^2$, $(0 \leq \alpha < \infty$, $\lambda = -s, \ldots, s$, $s =$ spin) of the first $C_1 = \mathbf{N}^2 - \mathbf{J}^2$, ($\mathbf{N}$, $\mathbf{J}$ are boost and rotation generators) and the eigenvalues $\alpha\lambda$ of the second Casimir operator $C_2 = \mathbf{N}\cdot\mathbf{J}$ of the Lorentz group. The representations (α, λ) and $(-\alpha, -\lambda)$ are unitarily equivalent. In the momentum representation ($\mathbf{p}=$ momentum, $p_0 = \sqrt{m^2c^4 + c^2\mathbf{p}^2}$, $m =$ mass, $s = 0$, $\mathbf{n}^2(\theta, \varphi) = 1$), the operator $C_1(\mathbf{p})$ has the eigenfunctions

$$\xi(\mathbf{p}, \alpha, \mathbf{n}) := \left[(p_0 - c\mathbf{p}\cdot\mathbf{n})/mc^2\right]^{-1+i\alpha}, \tag{1}$$

In [2], it was proposed to classify the states of a relativistic particle by means of the operators C_1 and C_2, and to carry out an expansion known as Shapiro transformations or the expansion of the Lorentz group

$$\psi(\alpha, \mathbf{n}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{p_0}\, \psi(\mathbf{p})\, \xi^*(\mathbf{p}, \alpha, \mathbf{n}). \tag{2}$$

Here $\psi(\mathbf{p})$ denotes the wave function of a spinless particle in the momentum space representation and $\psi(\alpha, \mathbf{n})$ the wave function of the particle in the $\alpha\mathbf{n}$ representation. The expansion (2) does not include any dependence on the time and the space coordinates t, $\mathbf{x}$, i.e. it is "spacetime independent". In [3,4] the expansion over the functions $\xi^*(\mathbf{p}, \alpha, \mathbf{n})$ was used to introduce the "relativistic configurational" representation (in the following $\rho\mathbf{n}$-representation, $\rho = \alpha\hbar/mc$) in the framework of a two-particle equation of the quasipotential type. In this approach the variable ρ was interpreted as the relativistic generalization of a relative coordinate. In [3,4] it was shown that the corresponding operators of the Hamiltonian $H(\rho, \mathbf{n})$ and the 3-momentum $\mathbf{P}(\rho, \mathbf{n})$, defined on the functions $\xi^*(\mathbf{p}, \rho, \mathbf{n})$, have the form of the differential-difference operators [their explicit form shall be used in what follows; see (32) and (33)]. Various applications of the unitary representations of the Lorentz group and the $\rho\mathbf{n}$-representation can be found in the literature.

In our previous papers [5–10] it has been shown that the UIR of the Lorentz group and the expansion (2) may also be used in a so-called generalized Schrödinger picture in which the analogs of the Schrödinger operators of a particle are independent of both the time and the space coordinates. It was found that the operators $H(\rho, \mathbf{n})$, $\mathbf{P}(\rho, \mathbf{n})$, $\mathbf{J}(\mathbf{n})$, and $\mathbf{N}(\rho, \mathbf{n}) = \rho\mathbf{n} + (\mathbf{n}\times\mathbf{J} - \mathbf{J}\times\mathbf{n})/2mc$ satisfy the commutations relations of the Poincaré algebra in the $\rho\mathbf{n}$-representation. In the GS picture we have two spacetime independent representations of the Poincaré algebra; the $\mathbf{p}$ and the $\rho\mathbf{n}$-representation. This leads to another Fourier transformation than that in the standard quantum theory. For a free particle in the Minkowski spacetime the coordinates t, $\mathbf{x}$ may be introduced in the states with the help of the transformation

$$S(t, \mathbf{x}) = \exp[-i(tH - \mathbf{x}\cdot\mathbf{P})/\hbar], \tag{3}$$

where H and $\mathbf{P}$ are the Hamilton and momentum operators of the particle in the $\rho\mathbf{n}$ or in the $\mathbf{p}$-representation. Instead of (2) we obtain

$$\begin{aligned} S(t, \mathbf{x})\psi(\rho, \mathbf{n}) &= \psi(\rho, \mathbf{n}, t, \mathbf{x}) \\ &= \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{p_0}\, \psi(\mathbf{p}, t, \mathbf{x})\, \xi^{*(0)}(\mathbf{p}, \rho, \mathbf{n}), \end{aligned} \tag{4}$$

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where $\psi(\mathbf{p}, t, \mathbf{x}) = \psi(\mathbf{p}) \exp[-i x \cdot p/\hbar]$. It follows that in the GS picture the spacetime coordinates appear in the states in the $\rho\mathbf{n}$ and in the $\mathbf{p}$-representation. The use of the $\rho\mathbf{n}$-representation in this picture makes it possible to describe extended objects like strings.

For the functions $\psi(\rho, \mathbf{n}, t, \mathbf{x})$ and $\psi(\mathbf{p}, t, \mathbf{x})$ we have

$$i\hbar \frac{\partial}{\partial t}\psi = H\psi, \quad -i\hbar \frac{\partial}{\partial \mathbf{x}}\psi = \mathbf{P}\psi. \tag{5}$$

In these equations the operators H and $\mathbf{P}$ induce on the left-hand-side the operators of the Lie-algebra of Killing vectors of the Minkowski spacetime. A full Poincaré algebra contain in addition to the operators in (5) the Lorentz rotation and boost generators. For the first Casimir operator of the Poincaré group we have

$$\begin{aligned} &C(t, \mathbf{x})\psi(\mathbf{p}, t, \mathbf{x}) = C(\mathbf{p})\psi(\mathbf{p}, t, \mathbf{x}), \\ &\quad C(t, \mathbf{x})\psi(\rho, \mathbf{n}, t, \mathbf{x}) = C(\rho\mathbf{n})\psi(\rho, \mathbf{n}, t, \mathbf{x}). \end{aligned} \tag{6}$$

In the paper [11], it was shown that equations like (5) may be used to describe a particle in an Anti-de Sitter spacetime. It was found that operators with an external field in the $\rho\mathbf{n}$-representation which corresponds to an attractive force induce operators $K_a(t, x^i)$ which are related to Killing vectors of the AdS spacetime ($a = 1, 2, \ldots 10$; $\{x^i\}$, $i = 1, 2, 3$). We have

$$K_a(t, x^i)\widetilde{\Phi}(\rho, \mathbf{n}, t, x^i) = B_a(\rho, \mathbf{n})\widetilde{\Phi}(\rho, \mathbf{n}, t, x^i). \tag{7}$$

Here $\widetilde{\Phi}$ denotes the wave function of the particle in the $\rho\mathbf{n}$-representation. The operators $K_a(t, x^i)$ satisfy the same commutation rules as the spacetime independent operators $B_a(\rho, \mathbf{n})$, except for the minus signs on the right-hand sides. In the framework of the GS picture, a particle in a spacetime is free if a set of operators in the $\rho\mathbf{n}$ or in the $\mathbf{p}$-representation force the introduction of Killing vector fields of this spacetime.

In the present paper we will show that an external massless field in the $\rho\mathbf{n}$ or in the momentum representation which corresponds to a repulsive force induce operators which are related to Killing vectors of the de Sitter spacetime. We want to show that a particle in a de Sitter spacetime may be described by the equations like (7). The external fields in the $\mathbf{p}$ or in the $\rho\mathbf{n}$-representation do violate the commutation relations of the Poincaré algebra. We have the problem of determining observables in the GS picture. Here we use only the coordinate system which corresponds to an exponentially expanding world [12–14]. We shall show that in these coordinates the spacetime independent momentum operators of the particle in addition to the external field contain the Lorentz boost generators. In Sect. 2 we study the motion of a particle in a two dimensional de Sitter spacetime. We use the ρ-representation. In Sect. 3 the four dimensional de Sitter spacetime and the $\rho\mathbf{n}$-representation are used. In the appendix we use the momentum representation.

2 Motion in a two dimensional dS spacetime

In a two dimensional Minkowski spacetime the motion of a free particle is described by the equations

$$\begin{aligned} i\hbar \frac{\partial}{\partial t}\psi(\rho, t, x) &= H(\rho)\psi(\rho, t, x), \\ -i\hbar \frac{\partial}{\partial x}\psi(\rho, t, x) &= P(\rho)\psi(\rho, t, x), \end{aligned} \tag{8}$$

where the Hamilton operator $H(\rho)$ and the momentum operator $P(\rho)$ have the form

$$H(\rho) = mc^2 \cosh\left(-\frac{i\hbar}{mc}\partial_\rho\right), \; P(\rho) = mc \sinh\left(-\frac{i\hbar}{mc}\partial_\rho\right). \tag{9}$$

The operators $H(\rho)$, $P(\rho)$ and the operator $N(\rho)$ ($N(\rho) = \rho$) satisfy the commutation relations of the Poincaré algebra,

$$[N, P] = i\frac{\hbar}{mc^2}H, \quad [P, H] = 0, \quad [H, N] = -i\frac{\hbar}{m}P. \tag{10}$$

In (6), in a model of a relativistic oscillator we have used an external field which corresponds to an attractive force (ω = frequency)

$$\begin{aligned} \widetilde{H}'(\rho) &= \frac{m\omega^2}{2}\rho\left(\rho - i\frac{\hbar}{mc}\right)\mathrm{e}^{-i\frac{\hbar}{mc}\partial_\rho}, \\ \widetilde{P}_1'(\rho) &= \frac{m\omega^2}{2c}\rho\left(\rho - i\frac{\hbar}{mc}\right)\mathrm{e}^{-i\frac{\hbar}{mc}\partial_\rho}. \end{aligned} \tag{11}$$

The operators

$$\hat{P}_0(\rho) = H(\rho) + \widetilde{H}'(\rho), \quad \hat{P}_1(\rho) = P(\rho) + \widetilde{P}_1'(\rho), \tag{12}$$

and ρ satisfy the commutation relations

$$\begin{aligned} &[\rho, \hat{P}_1] = i\frac{\hbar}{mc^2}\hat{P}_0, \quad [\hat{P}_1, \hat{P}_0] = -i\hbar m\omega^2\rho, \\ &\quad [\hat{P}_0, \rho] = -i\frac{\hbar}{m}\hat{P}_1. \end{aligned} \tag{13}$$

In the nonrelativistic limit the operators $\hat{P}_0(\rho) - mc^2$ and $\hat{P}_1(\rho)$ assume the form

$$\hat{P}_{0\mathrm{nr}} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\rho^2} + \frac{m\omega^2}{2}\rho^2, \quad \hat{P}_{1\mathrm{nr}} = -i\hbar\frac{\partial}{\partial\rho}. \tag{14}$$

It was found that the operators $\hat{P}_0$, $\hat{P}_1(\rho)$, and ρ induce operators which are related to the Killing vectors of an AdS spacetime.

Here, in analogy to (11) for the external field which corresponds to a repulsive force we use the following operators

$$
\begin{aligned}
H_0^{'}(\rho) &= -\frac{mc^2}{2\ell^2}\rho\left(\rho - i\frac{\hbar}{mc}\right)\mathrm{e}^{-i\frac{\hbar}{mc}\partial_\rho},\\
P_1^{'}(\rho) &= -\frac{mc}{2\ell^2}\rho\left(\rho - i\frac{\hbar}{mc}\right)\mathrm{e}^{-i\frac{\hbar}{mc}\partial_\rho}
\end{aligned}
\tag{15}
$$

where the constant ℓ has the dimension of length. Here the quantity ℓ is related to the radius of a two dimensional de Sitter spacetime.

The operators

$$
\Pi_0(\rho) = H(\rho) + H_0^{'}(\rho), \quad \Pi_1(\rho) = P(\rho) + P_1^{'}(\rho), \tag{16}
$$

satisfy the commutation relations

$$
\begin{aligned}
&[\rho, \Pi_1] = i\frac{\hbar}{mc^2}\Pi_0, \quad [\Pi_1, \Pi_0] = i\hbar m\frac{c^2}{\ell^2}\rho,\\
&[\Pi_0, \rho] = -i\frac{\hbar}{m}\Pi_1.
\end{aligned}
\tag{17}
$$

The Casimir operator is a multiple of the identity operator I

$$
\begin{aligned}
C(\rho) &= -\frac{\ell^2}{(\hbar c)^2}\{\Pi_0{}^2 - c^2\Pi_1^2\} - \frac{m^2c^2}{\hbar^2}\rho^2\\
&= -\left(\frac{m^2c^2\ell^2}{\hbar^2}\right)I.
\end{aligned}
\tag{18}
$$

Now in accordance with (7), we introduce three operators K_{10}, K_{12}, and K_{02} in terms of the spacetime coordinates of a two dimensional de Sitter spacetime with the same commutation rules as the operators Π_0, Π_1, and ρ, except for the minus signs on the right-hand sides ($d = 2$)

$$
\begin{aligned}
&[K_{10}, K_{1d}] = -\frac{\imath\hbar}{mc^2}K_{0d}, \quad [K_{1d}, K_{0d}] = -\imath\hbar m\frac{c^2}{\ell^2}K_{10},\\
&[K_{0d}, K_{10}] = \frac{\imath\hbar}{m}K_{1d}.
\end{aligned}
\tag{19}
$$

For the operators K_{0d}, K_{1d}, and K_{10} we choose the following realization:

$$
K_{0d}(t,x) = i\hbar\left(\frac{\partial}{\partial t} - \frac{cx}{\ell}\frac{\partial}{\partial x}\right), \tag{20}
$$

$$
K_{1d}(t,x) = i\frac{\hbar}{\ell}\left[x\frac{\partial}{\partial ct} - \left(\frac{x^2}{2\ell} + \frac{\ell}{2}\mathrm{e}^{-2ct/\ell}\right)\frac{\partial}{\partial x} - \frac{\ell}{2}\frac{\partial}{\partial x}\right], \tag{21}
$$

$$
K_{01}(t,x) = i\frac{\hbar}{mc}\left[-x\frac{\partial}{\partial ct} + \left(\frac{x^2}{2\ell} + \frac{\ell}{2}\mathrm{e}^{-2ct/\ell}\right)\frac{\partial}{\partial x} - \frac{\ell}{2}\frac{\partial}{\partial x}\right]. \tag{22}
$$

These operators are related to the Killing vectors of the de Sitter spacetime with metric

$$
ds^2 = c^2dt^2 - \mathrm{e}^{2ct/\ell}dx^2. \tag{23}
$$

Using (17) and (19) we find that a free particle in dS spacetime (23) is describe by the quantum equations

$$
\begin{aligned}
&K_{0d}(t,x)\Phi(\rho,t,x) = \Pi_0\Phi(\rho,t,x),\\
&K_{1d}(t,x)\Phi(\rho,t,x) = \Pi_1\Phi(\rho,t,x), \qquad (24)\\
&K_{10}(t,x)\Phi(\rho,t,x) = \rho\Phi(\rho,t,x). \qquad (25)
\end{aligned}
$$

Here $\Phi(\rho, t, x)$ denotes the wave function of the particle.

For the Casimir operators $C(\rho)$ and

$$
C(t,x) = \frac{l^2}{c^2}\frac{\partial^2}{\partial t^2} + \frac{l}{c}\frac{\partial}{\partial t} - l^2\mathrm{e}^{-2ct/l}\frac{\partial^2}{\partial x^2} \tag{26}
$$

we have

$$
C(t,x)\Phi(\rho;t,x) = C(\rho)\Phi(\rho;t,x). \tag{27}
$$

To proceed further, we must construct a quantum equation of the particle with the property given in (5a) or in (5b) above. The explicit forms of the operators $K_{0d}(t, x)$, $K_{1d}(t, x)$ show that the operators $\Pi_0(\rho)$ and $\Pi_1(\rho)$ in (24) cannot be defined as the Hamilton and the momentum operators of the particle. In the metric (23) we can construct a momentum operator by using two sums $K_{1d} + \frac{mc}{\ell}K_{01}$, and $\hat{\Pi}_1(\rho) + \frac{mc}{\ell}\rho$. We obtain the equation

$$
-i\hbar\frac{\partial}{\partial x}\Phi(\rho;t,x) = \left[\Pi_1(\rho) + \frac{mc}{\ell}\rho\right]\Phi(\rho;t,x), \tag{28}
$$

which defines the operator on the right-hand side as the momentum operator of the particle in the ρ-representation. This operator in addition to the external field contains the operator ρ multiplied by the mass m and the parameter $\frac{c}{\ell}$. For the eigenfunctions $v_\kappa(\rho)$ of the operator $\Pi_1(\rho) + \frac{mc}{\ell}\rho$ we have

$$
\left(\Pi_1(\rho) + \frac{mc}{\ell}\rho\right)v_\kappa(\rho) = \kappa v_\kappa(\rho), \tag{29}
$$

where κ denotes the value of the momentum of the particle in the de Sitter spacetime. A general solution of $\Phi(\rho; t, x)$ can be written in the form

$$
\Phi(\rho;t,x) = \int v_\kappa(\rho)f_\kappa(t,x)d\kappa, \tag{30}
$$

where $f_\kappa(t, x)$ are the eigenfunctions of the operators $C(t, x)$.

For the propagator of the particle we have the expression

$$
\mathcal{K}(2,1) = \int v_\kappa(\rho_2)f_\kappa(t_2,x_2)v_\kappa^*(\rho_1)f_\kappa^*(t_1,x_1)d\kappa. \tag{31}
$$

3 Motion in a four dimensional dS spacetime

The operators $H(\rho,\mathbf{n})$ and $\mathbf{P}(\rho,\mathbf{n})$ in (5) have the form (spin = 0)

$$H(\rho,\mathbf{n}) = mc^2\cosh\left(\frac{i\hbar}{mc}\partial_\rho\right) + \frac{i\hbar c}{\rho}\sinh\left(\frac{i\hbar}{mc}\partial_\rho\right) + \frac{\mathbf{L}^2(\mathbf{n})}{2m\rho^2}\mathrm{e}^{\frac{i\hbar}{mc}\partial_\rho}, \tag{32}$$

$$\mathbf{P} = \mathbf{n}H/c - \frac{mc}{\rho}\mathrm{e}^{\frac{i\hbar}{mc}\partial_\rho}\mathbf{N}(\rho,\mathbf{n}). \tag{33}$$

Here $\mathbf{L}(\mathbf{n})$ and $\mathbf{N}(\rho,\mathbf{n}) = \rho\mathbf{n} + (\mathbf{n}\times\mathbf{L} - \mathbf{L}\times\mathbf{n})/2mc$ are the operators of the Lorentz algebra in the $\rho\mathbf{n}$-representation. For a particle in an external field which corresponds to a repulsive force we use the operators

$$\Pi_0(\rho,\mathbf{n}) = H(\rho,\mathbf{n}) + H_0'(\rho,\mathbf{n}), \tag{34}$$
$$\Pi_i(\rho,\mathbf{n}) = P_i(\rho,\mathbf{n}) + P_i'(\rho,\mathbf{n}), \tag{35}$$

where

$$H_0' = -\frac{mc^2}{2\ell^2}\left(\rho - i\frac{\hbar}{mc}\right)^2 \mathrm{e}^{-i\frac{\hbar}{mc}\partial_\rho},$$
$$P_i' = -n_i\frac{mc}{2\ell^2}\left(\rho - i\frac{\hbar}{mc}\right)^2 \mathrm{e}^{-i\frac{\hbar}{mc}\partial_\rho}. \tag{36}$$

Here the quantity ℓ is related to the radius of a four dimensional de Sitter spacetime.

We have

$$H_0'^2 - c^2\mathbf{P}'^2 = 0. \tag{37}$$

Thus we see that the external field corresponds to a massless field. The operators H_0', $\mathbf{P}'$, $\mathbf{L}(\mathbf{n})$, and $\mathbf{N}(\rho,\mathbf{n})$ satisfy the commutations relations of the Poincaré algebra.

For the operators $\Pi_0(\rho,\mathbf{n})$, $\Pi_i(\rho,\mathbf{n})$ and $\mathbf{L}(\mathbf{n})$, $\mathbf{N}(\rho,\mathbf{n})$ we have

$$[N_i,\Pi_j] = \frac{\imath\hbar}{mc^2}\delta_{ij}\Pi_0, \quad [\Pi_i,\Pi_0] = \imath\hbar m\frac{c^2}{\ell^2}N_i$$
$$[\Pi_0,N_i] = -\frac{\imath\hbar}{m}\Pi_i, \tag{38}$$

$$[\Pi_i,\Pi_j] = \imath\frac{\hbar}{\ell^2}\epsilon_{ijk}L_k, \quad [L_i,\Pi_0] = 0,$$
$$[\Pi_i,L_j] = \imath\hbar\epsilon_{ijk}\Pi_k, \tag{39}$$

$$[L_i,L_j] = \imath\hbar\epsilon_{ijk}L_k, \quad [N_i,N_j] = -\frac{\imath\hbar}{m^2c^2}\epsilon_{ijk}L_k,$$
$$[N_i,L_j] = \imath\hbar\epsilon_{ijk}N_k. \tag{40}$$

The operators $\{\Pi_0,\Pi_i,N_i,L_i\}$ form a basis for the $\rho\mathbf{n}$-representation of the $SO(1,4)$ group generators and correspond to constants of motion. The Casimir operator

$$C(\rho,\mathbf{n}) = -\frac{\ell^2}{(\hbar c)^2}\left\{\Pi_0{}^2 - c^2\sum_{i=1}^3\Pi_i^2\right\} - \frac{m^2c^2}{\hbar^2}\mathbf{N}^2 + \frac{1}{\hbar^2}\mathbf{L}^2 \tag{41}$$

is a multiple of the identity operator

$$C(\rho,\mathbf{n}) = \left(-\frac{m^2c^2\ell^2}{\hbar^2} - 2\right)I. \tag{42}$$

In the four dimensional dS spacetime we use the metric which corresponds to an exponentially expanding world

$$ds^2 = c^2dt^2 - \mathrm{e}^{2ct/\ell}dx_i dx^i. \tag{43}$$

Here the quantity ℓ is related to the radius of the four dimensional de Sitter spacetime.

Now we introduce ten operators which are related to the Killing vectors of the spacetime (43), ($r^2 = (x^i)^2$)

$$K_{04} = i\hbar\left(\frac{\partial}{\partial t} - \frac{cx^i}{\ell}\frac{\partial}{\partial x^i}\right), \tag{44}$$

$$K_{i4} = i\frac{\hbar}{\ell}\left[x_i\left(\frac{\partial}{\partial ct} - \frac{x^i}{\ell}\frac{\partial}{\partial x^i}\right) - \frac{\ell}{2}\frac{\partial}{\partial x_i} - \left(\frac{\ell}{2}\mathrm{e}^{-2ct/\ell} - \frac{r^2}{2\ell}\right)\frac{\partial}{\partial x_i}\right], \tag{45}$$

$$K_{0i} = i\frac{\hbar}{mc}\left[-x_i\left(\frac{\partial}{\partial ct} - \frac{x^i}{\ell}\frac{\partial}{\partial x^i}\right) - \frac{\ell}{2}\frac{\partial}{\partial x_i} + \left(\frac{\ell}{2}\mathrm{e}^{-2ct/\ell} - \frac{r^2}{2\ell}\right)\frac{\partial}{\partial x_i}\right], \tag{46}$$

$$K_{ij} = i\hbar\left(x_i\frac{\partial}{\partial x^j} - x_j\frac{\partial}{\partial x^i}\right). \tag{47}$$

This operators satisfy the same commutation rules as the spacetime independent operators $\Pi_0(\rho,\mathbf{n})$, $\Pi_i(\rho,\mathbf{n})$ and $\mathbf{L}(\mathbf{n})$, $\mathbf{N}(\rho,\mathbf{n})$, except for the minus signs on the right-hand sides ($d = 4$),

$$[K_{i0},K_{jd}] = -\frac{\imath\hbar}{mc^2}\delta_{ij}K_{0d},$$
$$[K_{id},K_{0d}] = -\imath\hbar m\frac{c^2}{\ell^2}K_{i0}, \tag{48}$$

$$[K_{0d},K_{i0}] = \frac{\imath\hbar}{m}K_{i4} \quad [K_{id},K_{jd}] = -\imath\frac{\hbar}{\ell^2}K_{ij},$$
$$[K_{ij},K_{0d}] = 0, \tag{49}$$

$$[K_{id},K_{ik}] = \imath\hbar K_{kd}, \quad [K_{i0},K_{j0}] = \frac{\imath\hbar}{m^2c^2}K_{ij},$$
$$[K_{i0},K_{ik}] = \imath\hbar K_{k0}. \tag{50}$$

Just as we did in (24) and (25), we write ($\Phi = \Phi(\rho, \mathbf{n}, t, x^i)$)

$$K_{0d}(t, x^i)\Phi = \Pi_0(\rho, \mathbf{n})\Phi, \quad K_{id}(t, x^i)\Phi = \Pi_i(\rho, \mathbf{n})\Phi, \tag{51}$$

$$K_{i0}(t, x^i)\Phi = N_i(\rho, \mathbf{n})\Phi, \quad K_{ij}(x^i)\Phi = \varepsilon_{ijk}L_k(\mathbf{n})\Phi. \tag{52}$$

These quantum equations may be used to describe a free particle in de Sitter spacetime (43).

For the Casimir operators we obtain

$$C(t, x^i)\Phi = C((\rho, \mathbf{n})\Phi = \left(-\frac{m^2c^2\ell^2}{\hbar^2} - 2\right)\Phi \tag{53}$$

where

$$C(t, x^i) = \frac{\ell^2}{c^2}\frac{\partial^2}{\partial t^2} + 3\frac{\ell}{c}\frac{\partial}{\partial t} - \ell^2 \mathrm{e}^{-2\nu t}\nabla^2. \tag{54}$$

Now we have a situation fully analogous to that described in the previous section. From the equation

$$i\hbar\left(\frac{\partial}{\partial t} - \frac{cx^i}{\ell}\frac{\partial}{\partial x^i}\right)\Phi(\rho, \mathbf{n}, t, x^i) = \Pi_0(\rho, \mathbf{n})\Phi(\rho, \mathbf{n}, t, x^i) \tag{55}$$

follows that the operator $\Pi_0(\rho, \mathbf{n})$ cannot be defined as the Hamilton operator of the particle. In this metric we can again construct the momentum operators by using two sums,

$$K_{i4} + \frac{mc}{\ell}K_{0i} = \Pi_i + \frac{mc}{\ell}N_i. \tag{56}$$

We obtain

$$-i\hbar\frac{\partial}{\partial x_i}\Phi(\rho, \mathbf{n}, t, x^i) = \left[\Pi_i(\rho, \mathbf{n}). + \frac{mc}{\ell}N_i(\rho, \mathbf{n})\right] \times \Phi(\rho, \mathbf{n}, t, x^i). \tag{57}$$

Equation (57) defines the operators of the right-hand sides as the momentum operators of the particle in the $\rho\mathbf{n}$-representation. These operators in addition to the external field contain the Lorentz boost generators $N_i(\rho, \mathbf{n})$ multiplied by the mass m and the Hubble parameter $\frac{c}{\ell}$.

A general solution of Φ can be written in the form ($\boldsymbol{\kappa}$ = momentum)

$$\Phi(\rho, \mathbf{n}, t, x^i) = \int v_{\boldsymbol{\kappa}}(\rho, \mathbf{n}) f_{\boldsymbol{\kappa}}(t, x^i) d\kappa_1 d\kappa_2 d\kappa_3 \tag{58}$$

where $v_{\boldsymbol{\kappa}}(\rho, \mathbf{n})$ satisfy the equation

$$\sum_{i=1}^{3}\left(\Pi_i + \frac{mc}{\ell}N_i\right)^2 v_{\boldsymbol{\kappa}}(\rho, \mathbf{n}) = \boldsymbol{\kappa}^2 v_{\boldsymbol{\kappa}}(\rho, \mathbf{n}), \tag{59}$$

and $f_{\boldsymbol{\kappa}}(t, x^i)$ are the eigenfunctions of the Casimir operator $C(t, x^i)$.

4 Conclusion

In this paper we have shown that a generalized Schrödinger picture may be used to describe a free particle in a de Sitter spacetime. In this picture the analogs of the Schrödinger operators of a particle are independent of both the time and the space coordinates. These operators force the introduction of operators of the Killing vectors of a spacetime. The presence of this vector fields in a quantum equation determines the motion of a free particle in the corresponding spacetime. It was found that the spacetime independent operators with an external massless field which corresponds to a *repulsive force* induce the operators of Killing vectors of the de Sitter spacetime. The problem of determining the observables is based on choosing of the coordinate system in this spacetime. We have shown that in the coordinates which correspond to an exponentially expanding world the spacetime independent momentum operators of the particle in addition to the external field contain the Lorentz boost generators miltipled by the mass of the particle and the Hubble parameter. We hope that the formalism that has been presented here will be employed for solving problems in relativistic quantum physics, astrophysics, and cosmology.

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5 Appendix

In a two dimensional Minkowski spacetime a free relativistic particle is described by (5). In the momentum representation the Hamilton and the momentum operator may be written in the form ($\chi = -\ln[(p_0 - cp)/mc^2]$)

$$p_0 = mc^2\cosh\chi, \quad p = mc\sinh\chi. \tag{60}$$

For a particle in an external field which corresponds to a repulsive force we use the operators

$$\Pi_0(\chi) = mc^2\left[\cosh\chi + \frac{1}{2}\left(\frac{\hbar}{mc\ell}\right)^2 e^{\chi}\left\{\frac{d^2}{d\chi^2} + \frac{d}{d\chi}\right\}\right], \tag{61}$$

$$\Pi_1(\chi) = mc\left[\sinh\chi + \frac{1}{2}\left(\frac{\hbar}{mc\ell}\right)^2 e^{\chi}\left\{\frac{d^2}{d\chi^2} + \frac{d}{d\chi}\right\}\right], \tag{62}$$

and $N(\chi) = i\frac{\hbar}{mc}\frac{\partial}{\partial\chi}$. They satisfy the commutation relations

$$[N(\chi), \Pi_1(\chi)] = i\frac{\hbar}{mc^2}\Pi_0(\chi),$$

$$[\Pi_1(\chi), \Pi_0(\chi)] = i\hbar\frac{mc^2}{l^2}N(\chi), \tag{63}$$

$$[\Pi_0(\chi), N(\chi)] = -i\frac{\hbar}{m}\Pi_1(\chi). \tag{64}$$

We have

$$K_{d0}(t,x)\Phi(\chi;t,x) = \Pi_0(\chi)\Phi(\chi;t,x), \tag{65}$$

$$K_{1d}(t,x)\Phi(\chi;t,x) = \Pi_1(\chi)\Phi(\chi;t,x),$$

$$K_{01}(t,x)\Phi(\chi;t,x) = N(\chi)\Phi(\chi;t,x). \tag{66}$$

where the function $\Phi(\chi;t,x)$ describe a particle in two dimensional de Sitter spacetime with metric (23).

Now, from the operators $K_{1d} + \frac{mc}{\ell}K_{01}$, and $\hat{\Pi}_1 + \frac{mc}{\ell}N$ we obtain the equation

$$-i\hbar\frac{\partial}{\partial x}\Phi(\chi;t,x) = \left(\hat{\Pi}_1(\chi) + \frac{mc}{\ell}N(\chi)\right)\Phi(\chi;t,x) \tag{67}$$

which defines the operator $\hat{\Pi}_1(\chi) + \frac{mc}{\ell}N(\chi)$ as the momentum operator of the particle in the χ-representation.

The function $\Phi(\chi;t,x)$ can be written in the form

$$\Phi(\chi;t,x) = \int v_\kappa(\chi)f_\kappa(t,x)d\kappa, \tag{68}$$

where $v_\kappa(\chi)$ are the eigenfunctions of the operator $\hat{\Pi}_1(\chi) + \frac{mc}{\ell}N(\chi)$.

References

1. H. Joos, Fortschritte der Physik **10**, 65 (1962)
2. I.S. Shapiro, Sov. Phys. Doklady. **1**, 91 (1956)
3. V.G. Kadyshevsky, R.M. Mir-Kasimov, N.B. Skachkov, Nuovo Cimento A **55**, 233 (1968)
4. V.G. Kadyshevsky, R.M. Mir-Kasimov, N.B. Skachkov, Sov. J. Part. Nucl. **2**, 69 (1973)
5. R.A. Frick, Sov. J. Nucl. Phys. **38**, 481 (1983)
6. R.A. Frick, JETP Lett. **39**, 89 (1984)
7. R.A. Frick, J. Math. Phys. **38**, 3457 (1997)
8. R.A. Frick, Eur. Phys. J. C **22**, 581 (2001)
9. R.A. Frick, Eur. Phys. J. C **28**, 431 (2003)
10. R.A. Frick, Ann. Phys. (Berlin) **17**, 937 (2008)
11. R.A. Frick, Ann. Phys. (Berlin) **523**, 871 (2011)
12. N.D. Birrell, P.C.W. Davies, *Quantum fields in curved space* (Cambridge University Press, Cambridge, 1982)
13. M. Spradlin, A. Strominger, A. Volovich, Les Houches lectures on de Sitter space. arXiv:hep-th/0110007
14. Y.N. Obukhov, D. Puetzfeld, Phys. Rev. **D83**, 044024 (2011)